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## *On the Calculation of Life Contingencies.** (*Part II.*) *By* PROFESSOR DE MORGAN.

THE present paper is a continuation of that in the Companion for 1840, on the application of Barrett's method of calculating life contingencies. No tables were then published for contingencies involving two lives; but since that time Mr. Jones's tables have appeared, in his work on the subject in the Library of Useful Knowledge. It may therefore be worth while to collect and describe the formulæ relative to two lives which are most likely to be useful in practice; with some other matters, which may partly interest the actuary and partly the commencing student.

The method described in the former paper is extended by calculating every case of

$$D_{m,n}=a_m a_n v^k,\ D_{m+1,\,n+1}=a_{m+1}a_{n+1}v^{k+1},\ \&c.$$

where $a_x$ always means the number living at the age $x$, $v^x$ the present value of £1 due in $x$ years, and $k$ is any convenient number depending on $m$ or $n$, or both. In the tables published, $k$ is taken to be the greater of the two, $m$ or $n$; the consequence of which is, that two formulæ are necessary in every case in which the order of survivorship is a part of the question, according as the life, on the survivorship of which a benefit depends, is the elder or younger of

* See Note, p. 328, vol. xii. This part appeared in the *Companion to the Almanac* for 1842.

the two. This might have been avoided by choosing $\frac{1}{2}(m+n)$ for $k$, and on such an assumption we shall first exhibit the formulæ which the different cases require; pointing out, at the same time, how to adapt the results to the published tables. Such a course will not only be the most simple with respect to the theory, but will always enable the reader to detect a misprint, if any should exist.

We begin, then, as follows:—

$$D_{x,y}=a_x a_y v^{\frac{1}{2}(x+y)},\ N_{x,y}=D_{x+1,y+1}+D_{x+2,y+2}+\ \ldots\ldots$$

If the tables were complete we should then have

$$C_{x,y}=(a_x a_y-a_{x+1}a_{y+1})v^{\frac{1}{2}(x+y)+1}$$
$$M_{x,y}=C_{x,y}+C_{x+1,y+1}+\ \ldots\ldots$$

but as these tables have not been published, we must express M in terms of N as follows:—

$$M_{x,y}=vN_{x-1,y-1}-N_{x,y}=D_{x,y}-(1-v)N_{x-1,y-1}$$

In the former paper $N_{x,y}$ stood for $N_x-N_{x+y}$; in the present one let

$N_{x,y}-N_{x+z,y+z}$ be denoted* by $N_{x,y|z}$.

In all questions in which the joint duration of two lives takes the place of one life, the formulæ are obtained from those of the former paper by simply substituting $N_{x,y}$ for $N_x$, and so on. The following are instances, ($x$) and ($y$) representing individuals aged $x$ and $y$ years:—

1. The value of an annuity of £1 on the joint lives of ($x$) and ($y$) is

$N_{x,y}$ divided by $D_{x,y}$

2. The same, for $n$ years, if both parties live so long, is

$N_{x,y|n}$ or $N_{x,y}-N_{x+n,y+n}$, divided by $D_{x,y}$

3. The value of £1 to be received in $k$ years, if both ($x$) and ($y$) be then alive, is

$D_{x+k,y+k}$ divided by $D_{x,y}$

4. The value of an annuity of £1 deferred for $k$ years is

$N_{x+k,y+k}$ divided by $D_{x,y}$

5. The premium for the same ($k$ premiums in all) is

$N_{x+k,y+k}$ divided by $N_{x-1,y-1}-N_{x+k-1,y+k-1}$

Or, to pay for it in $l$ premiums, take $N_{x-1,y-1}-N_{x+l-1,y+l-1}$ for a divisor.

* It was not contemplated in the last paper that any one would have had the courage to compute tables for two lives. Had it been thought likely, what was there denoted by $N_{x,y}$ would have been denoted by $N_{x|y}$.

6. Present value of an assurance of £1 on the death of the first of $(x)$ and $(y)$.

$$\frac{M_{x,y}}{D_{x,y}} \text{ or } \frac{vN_{x-1,y-1}-N_{x,y}}{D_{x,y}} \text{ or } \frac{D_{x,y}-(1-v)N_{x-1,y-1}}{D_{x,y}}$$

7. The same, if the first death take place within $n$ years.

$$\frac{M_{x,y|n}}{D_{x,y}} \text{ or } \frac{v(N_{x-1,y-1}-N_{x+n-1,y+n-1})-(N_{x,y}-N_{x+n,y+n})}{D_{x,y}}$$

$$\text{or } \frac{D_{x,y}-D_{x+n,y+n}-(1-v)(N_{x-1,y-1}-N_{x+n-1,y+n-1})}{D_{x,y}}$$

8. To pay for either of the preceding in premiums, write for the divisor

$$N_{x-1,y-1} \text{ or } N_{x-1,y-1}-N_{x+l-1,y+l-1}$$

according as the premium is for life or for $l$ years.

These are the most common cases, and any others which arise can as easily be found from the preceding paper. The usual cases of reversionary annuities, annuities on the longest liver, &c., may be omitted here, as they are more easily calculated by the common tables.

In cases which involve the order of survivorship, tables in which $\frac{1}{2}(x+y)$ is the exponent of $v$ in $D_{x,y}$, will introduce the square root of $v$. The method of adapting their results to the case in which that exponent is the *greater of the two, x* or *y*, must be described as follows:—

Change $D_{x,y}$ into $D_{x,y}v^{-\frac{1}{2}(x-y)}$ or $D_{x,y}v^{-\frac{1}{2}(y-x)}$ according as $(x)$ or $(y)$ is the older life; and the same for $N_{x,y}$.

In the following instances the first result given is that in which $D_{x,y}$ stands for $a_x a_y v^{\frac{1}{2}(x+y)}$; the second, that in which $(x)$ is the older and $D_{x,y}$ denotes $a_x a_y v^x$; the third, that in which $(y)$ is the older and $D_{x,y}$ denotes $a_x a_y v^y$.

9. Required the present value and premium of an assurance of £1 on the life of $(x)$, provided that $(y)$ survive $(x)$. The first divisor is for the present value, the second for the premium.

$$\frac{vN_{x-1,y-1}-N_{x,y}+\sqrt{v}(N_{x-1,y}-N_{x,y-1})}{2D_{x,y} \text{ or } 2N_{x-1,y-1}}$$

$$(x>y) \quad \frac{v(N_{x-1,y-1}+N_{x-1,y})-(N_{x,y-1}+N_{x,y})}{2D_{x,y} \text{ or } 2N_{x-1,y-1}}$$

$$(y>x) \quad \frac{v(N_{x-1,y-1}-N_{x,y-1})+N_{x-1,y}-N_{x,y}}{2D_{x,y} \text{ or } 2N_{x-1,y-1}}$$

$(y=x)$ $\frac{1}{2}$ result of (6)

10. If the risk of the preceding question be not to begin until $k$ years have elapsed, write $x+k$ and $y+k$ instead of $x$ and $y$ *in the numerators only*, which then become

$$(x>y) \quad v(N_{x+k-1,\,y+k-1}+N_{x+k-1,\,y+k})-(N_{x+k,\,y+k-1}+N_{x+k,\,y+k})$$

$$(y>x) \quad v(N_{x+k-1,\,y+k-1}-N_{x+k,\,y+k-1})+N_{x+k-1,\,y+k}-N_{x+k,\,y+k}$$

11. And in all cases in which premiums are to be paid $l$ times, instead of during the whole joint duration, the divisor has

$$N_{x-1,\,y-1|l} \text{ or } N_{x-1,\,y-1}-N_{x+l-1,\,y+l-1}, \text{ instead of } N_{x-1,\,y-1}.$$

It will very much simplify both the use and theory of these formulæ if the student make himself perfectly master of the symbol $N_{x,\,y|t}$, which means the value of N at the ages $x$ and $y$ cut off, as it were, at the end of $t$ years, or diminished by all the terms arising from the period beyond $t$ years. Thus

$$\begin{aligned} N_{x,\,y} &= D_{x+1,\,y+1}+\ldots\ldots+D_{y+t,\,y+t}+D_{x+t+1,\,y+t+1}+\ldots\ldots \\ N_{x+t,\,y+t} &= \qquad\qquad\qquad\qquad\qquad\quad D_{x+t+1,\,y+t+1}+\ldots\ldots \\ N_{x,\,y}-N_{x+t,\,y+t} &= D_{x+1,\,y+1}+\ldots\ldots+D_{x+t,\,y+t} \\ &= N_{x,\,y|t} \end{aligned}$$

12. Required the present value and premium (payable $l$ times) of £1 to be paid at the death of $(x)$, if within $n$ years, and if $(y)$ be then alive.

Cut off every term in the numerator of (9) at $n$ years, and use the divisor for $l$ premiums, when necessary:

$$(x>y) \quad \frac{v(N_{x-1,\,y-1|n}+N_{x-1,\,y|n})-(N_{x,\,y-1|n}+N_{x,\,y|n})}{2D_{x,\,y} \text{ or } 2N_{x-1,\,y-1|l}}$$

$$(y>x) \quad \frac{v(N_{x-1,\,y-1|n}-N_{x,\,y-1|n})+N_{x-1,\,y|n}-N_{x,\,y|n}}{2D_{x,\,y} \text{ or } 2N_{x-1,\,y-1|l}}$$

13. To find the corresponding formulæ in the case in which the assurance is for the death of $(x)$, provided $y$ be then *dead*, calculate the present value of the assurance on the life of $(x)$ absolutely (by the former paper), and subtract the present value of the proper one of the preceding formulæ; the remainder is the present value required. Find the premium by the usual method, or else by multiplying by $D_{x,\,y}$ and dividing by the proper divisor.

14. The present value of a life annuity of $n$ payments on $(y)$, the first being made at the $(k+1)$th anniversary of the present time which follows the death of $(x)$.

N.B. If $k=0$ and $n$ outrun the term of life, this is a common reversionary annuity on $(y)$ after $(x)$.

$$\frac{N_{y+k}-N_{y+k+n}}{D_y}-\frac{N_{x,\,y+k}v^{\frac{1}{2}k}-N_{x,\,y+k+n}v^{\frac{1}{2}(k+n)}}{D_{x,\,y}}$$

To suit the published tables the second numerator is to be altered as follows, all the rest remaining :—

| | | |
|---|---|---|
| $x <$ or $= y$ | into | $N_{x,\,y+k} - N_{x,\,y+k+n}$ |
| $x > y <$ or $= y+k$ | ,, | $(N_{x,\,y+k} - N_{x,\,y+k+n})v^{x-y}$ |
| $x > y+k <$ or $= y+k+n$ | ,, | $N_{x,\,y+k}v^{k} - N_{x,\,y+k+n}v^{x-y}$ |
| $x > y+k+n$ | ,, | $N_{x,\,y+k}v^{k} - N_{x,\,y+k+n}v^{k+n}$ |

15. The present value of £1 to be paid at the death of $(x)$, if he die before $(y)$ or within $n$ years after $(y)$.

$$\frac{M_x - M_{x+n}}{D_x} +$$

$$v^{\frac{1}{2}n}\,\frac{N_{x+n-1,\,y-1}v - N_{x+n,\,y} + v^{\frac{1}{2}}(N_{x+n-1,\,y} - N_{x+n,\,y-1})}{2D_{x,\,y}}$$

To suit the published tables the second numerator ($v^{\frac{1}{2}n}$ included) must be

$$(x+n<y)$$

$$(N_{x+n-1,\,y-1} - N_{x+n,\,y-1})v^{n+1} + (N_{x+n-1,\,y} - N_{x+n,\,y})v^{n}$$

$$(x<y,\ x+n>y)$$

$$\{(N_{x+n-1,\,y-1} + N_{x+n-1,\,y})v - (N_{x+n,\,y} + N_{x+n,\,y-1})\}v^{y-x}$$

$x > y$ $\quad (N_{x+n-1,\,y-1} + N_{x+n-1,\,y})v - (N_{x+n,\,y} + N_{x+n,\,y-1})$

If this risk be to continue only for $n+p$ years, all the Ns must be stopped after $p$ years, or $N_{x+n-1,\,y-1\,|\,p}$ must be written for $N_{x+n-1,\,y-1}$, and so on.

If $x+n=y$, the second numerator must be

$$\{(N_{x+n-1,\,y-1}v + N_{x+n-1,\,y}) - (N_{x+n,\,y} + N_{x+n,\,y-1})\}v^{n}$$

16. The present value of £1 to be paid on the death of $(x)$, provided he die within $n$ years after $y$.

From the preceding result subtract the present value of £1 payable at the death of $(x)$, if $(y)$ survive. (See No. 9.)

17. The present value of £(1) to be paid at the death of $(x)$, if he survive $(y)$.

From the present value of an assurance on the life of $(x)$ subtract the result of (9), or the value of £1 to be paid at his death, if $y$ survive.

18. The present value of £1 to be paid at the death of $(x)$, if more than $n$ years after the death of $(y)$.

From the present value of an assurance of £1 on the life of $(x)$ subtract the result of (15). This gives $M_{x+n} \div D_x$ for the first term.

I have not thought it necessary to write down the formulæ connected with the last few cases, as they seldom occur.

There are, we are told, still some persons who cannot see how much shorter this method of Barrett's is than that in previous use. It is certain that any new method does not become so easy as the old and well-known one until a few examples have been tried; and there are dispositions which will not give a new method a fair trial. But, for those who may be commencing the subject, and to whom, therefore, all methods stand entirely on their own merits, we subjoin the example of a simple temporary assurance on two joint lives, worked by both methods.

Required the present value of an assurance for the next ten years only, on the joint continuance of two lives, now aged 45, and 50 years: interest 5 per cent. (Milne, p. 341.)

*Old Method.*

| | | | |
|---|---|---|---|
| $a_{55}$ | 3·60991 | 1—·43827 | |
| $a_{60}$ | 3·56146 | =·56173 | $\bar{1}$·74953 |
| $a^{-1}_{45}$ | $\bar{4}$·32541 | $v$ | $\bar{1}$·97881 |
| $a^{-1}_{50}$ | $\bar{4}$·35684 | | |
| $v^{10}$ | $\bar{1}$·78811 | ·53498 | $\bar{1}$·72834 |
| ·43827 | $\bar{1}$·64173 | 6·623 | ·82105 |
| | ·85163 | $1-v$ | $\bar{2}$·67778 |
| 3·114 | ·49336 | ·31538 | $\bar{1}$·49883 |
| 9·737 | | ·53498 | |
| 6·623 | | ·21960 present value of £1. | |

*Barrett's Method.*

| | | | | | |
|---|---|---|---|---|---|
| $N_{44,49}$ | 19460646 | | $N_{45,50}$ | 17648150 | |
| $N_{54,59}$ | 6438764 | | $N_{55,60}$ | 5644407 | |
| | 13021882 | 7·11468 | | 12003743 | |
| | $1-v$ | $\bar{2}$·67778 | | | |
| | 620100 | 5·79246 | | | |
| | 12401782 | | | | |
| | 12003743 | | | | |
| | 398039 | 5·59992 | | | 5·59992 |
| $D_{45,50}$ | 1812496 | 6·25828 | | 13021882 | 7·11467 |
| Present value | ·21960 | $\bar{1}$·34164 | Premium | ·030567 | $\bar{2}$·48525 |
| | | | Milne, p. 346, | ·030565 | |

If we were to look at the number of figures only there would seem to be little difference between the two methods, the first

containing about 130 figures and the second 120. But if we look at the number of entries of different tables in the two, we find it as follows:—

| *Old Method.* | | *New Method.* | |
|---|---|---|---|
| No. of pages in Milne which must be consulted for $\log a_{55}$, &c., $v^{10}$, 9·737, $v$ and $1-v$ | 3 | No. of pages in Jones to be consulted for $N_{44,59}$.... ..$D_{45,50}$ . . . . . . | 1 |
| Logarithms to be taken out . | 6 | Logarithms taken out . . . | 4 |
| Numbers to logarithms . . | 4 | Numbers to logarithms . . . | 2 |
| | 13 | | 7 |

In multiplying by $v$ in the above process, the formula taken is

$$av = a - (1-v)a,$$

and $(1-v)a$ is found by logarithms and subtracted from $a$. The reason is that $v$ is so near unity that it would take tables of logarithms to seven places to get five places of the answer correct, if $av$ were to be directly found.

As another example, take the following (Milne, p. 360): Required (at 5 per cent.) the annual premium for £1, payable at the death of (50), if within 10 years, and (45) be then alive; premiums being payable for ten years, if the joint lives endure so long. The first formula of No. 12 must be employed:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $N_{49,44}$ | 19460646 | $N_{50,44}$ | 18016514 | $N_{49,45}$ | 19057899 | $N_{50,45}$ | 17648150 |
| $N_{59,54}$ | 6438763 | $N_{60,54}$ | 5834103 | $N_{59,55}$ | 6227029 | $N_{60,55}$ | 5644407 |
| | 13021883 | | 12182411 | | 12830870 | | 12003743 |
| | 12830870 | | | | | | 12182411 |
| | 25852753 | | | | | | 24186154 |

| | |
|---|---|
| 25852753 | 7·4125067 |
| $1-v$ | $\bar{2}$·6777803 |
| 1231082 | 6·0902870 |
| 24621671 | |
| 24186154 | |
| 435517 | 5·6390051 |
| 13021883 | 7·1146737 |
| 2)·03344502 | $\bar{2}$·5243314 |
| ·01672251 required premium. | |

Let the problem be now changed by simply inverting the ages. Required the annual premium for £1 payable at the death of (45) if within ten years, and (50) be then alive. The second formula

of No. 12 must now be used, but our preceding process contains all the data:

| | | | |
|---|---|---|---|
| $N_{44,49\,|\,10}$ | 13021883 | $N_{44,50\,|\,10}$ | 12182411 |
| $N_{45,49\,|\,10}$ | 12830870 | $N_{45,50\,|\,10}$ | 12003743 |
| | 191013 | 5·2810629 | 178668 |
| | $v$ | $\bar{1}$·9788107 | |
| | 181917 | 5·2598736 | |
| | 178668 | | |
| | 360585 | 5·5570076 | |
| | 13021883 | 7·1146737 | |
| | 2)·02769069 | $\bar{2}$·4423339 | |
| | ·01384534 | present value required. | |
| | ·01672251 | result of the preceding. | |
| | ·03056785 | | |

This last result ·03056785 is the premium for the assurance on the joint lives, and agrees both with the above result and with Mr. Milne. But Mr. Milne, instead of ·01672251, has (p. 361) ·015714, differing materially from the truth. The reason is, as those who are used to the common method are well aware, that the old tables of annuities, calculated to three decimal places, are not sufficiently exact for survivorship questions, the results of which depend upon the differences of nearly equal annuities, particularly where, owing to the differences of ages given being only every five years, the defective interpolation by first differences is employed. Mr. Milne's process, it is hardly necessary to say, is as good as, with his tables, it could have been: but the above examples show, not only that Barrett's method is much the more easy of the two, but also the more exact, the tables actually published* being used. All the above results are true to six significant figures.

We now proceed to consider one or two questions in which approximations are usually made, with a view to state the results of an examination of the approximations, and to propose some amendments upon them.

1. The values of life annuities are usually calculated on the supposition that interest can be converted into principal only once a year, the truth being that such conversion always takes

* The superior exactness of the above method, as it stands, is not owing to anything in the method, but solely arises from the greater extent of the tables published.

place twice a year at least. Mr. Milne has given a rule (p. 367) for making the requisite correction, but I believe the following will be found to be more easy of application: it is given for the case of half-yearly conversions only. Let the value of an annuity (for lives or a term of years) be A at $c$ per cent., $A_1$ at $c+1$ per cent., $A_2$ at $c+2$ per cent., and so on. Take the differences of A, $A_1$, &c., as follows:—

$$\begin{array}{llll} A & & & \\ & B & & \\ A_1 & & C & \\ & B_1 & & D \\ A_2 & & C_1 & \\ & B_2 & & \\ A_3 & & & \end{array} \qquad \begin{array}{lll} B=A-A_1 & B_1=A_1-A_2 & B_2=A_2-A_3 \\ C=B-B_1 & C_1=B_1-B_2 & \\ D=C-C_1 & & \end{array}$$

Then the value of the same annuity at $c$ per cent., interest being convertible twice a year, is

$$A-\frac{c^2}{400}\left(B+\frac{C}{2}+\frac{D}{3}-\frac{c^2(C+D)}{800}\right), \text{ very nearly.}$$

Example 1. Required the value of a perpetual annuity of £1 at 4 per cent., payable yearly, on the supposition that money is convertible half yearly.

$$\begin{array}{llll} A\ =25{\cdot}000 & & & \\ & 5{\cdot}000 & & \\ A_1=20{\cdot}000 & & 1{\cdot}667 & \\ & 3{\cdot}333 & & {\cdot}715 \\ A_2=16{\cdot}667 & & {\cdot}952 & \\ & 2{\cdot}381 & & \\ A_3=14{\cdot}286 & & & \end{array}$$

$$(1{\cdot}667+{\cdot}715)\frac{16}{800}={\cdot}048 \qquad \begin{array}{r} 5{\cdot}000 \\ {\cdot}889 \\ {\cdot}238 \\ \hline 6{\cdot}127 \\ {\cdot}048 \\ \hline 6{\cdot}079 \end{array}$$

$$6{\cdot}079\times\frac{16}{400}={\cdot}243$$

$25{\cdot}000-{\cdot}243=24{\cdot}757$ answer.

The real answer is $24{\cdot}75225$, and a perpetual annuity is the severest trial which can be made of the rule.

Example 2. (Milne, p. 367.) Required the value of a life annuity on (60), at 5 per cent., interest being convertible twice a year:

$$\begin{array}{llll} A\ \ 8{\cdot}940 & & & \\ & {\cdot}636 & & \\ A_1\ \ 8{\cdot}304 & & {\cdot}075 & \\ & {\cdot}561 & & {\cdot}012 \\ A_2\ \ 7{\cdot}743 & & {\cdot}063 & \\ & {\cdot}498 & & \\ A_3\ \ 7{\cdot}245 & & & \end{array}$$

$$({\cdot}075+{\cdot}012)\frac{25}{800}={\cdot}0027 \qquad \begin{array}{r} {\cdot}6360 \\ {\cdot}0375 \\ {\cdot}0040 \\ \hline {\cdot}6775 \\ {\cdot}0027 \\ \hline {\cdot}6748 \end{array}$$

$${\cdot}6748\times\frac{25}{400}={\cdot}04218$$

$8{\cdot}940-{\cdot}04218=8{\cdot}89782$

Mr. Milne's answer is 8·8977.

2. When an annuity is to be paid, not yearly, but half-yearly or quarterly, &c., interest being convertible into principal only once a year, the common correction of A, the present value of a yearly annuity, is to write $A+\frac{n-1}{2n}$ instead of A, $n$ being the number of times which the annuity is payable yearly. This correction is strictly true when we speak of a perpetual annuity, on the supposition that by interest being convertible once a year, we mean that money may be invested at any one time of the year, so that the grantor of the annuity must be held to lose simple interest for the rest of the year on each of the prepaid portions. When this correction is to be investigated for a life annuity, it would generally be supposed that those who die in each year die at equal intervals. This is an insufficient supposition; for if the mortality of the table be increasing from year to year, it ought to be supposed that the mortality of the latter part of a year is greater than that of the former. We have therefore taken the supposition that the mortality of each year is to be constructed on the hypothesis of constant third differences, which has the effect of distributing the deaths of each year in a manner depending on the mortality of the preceding and following years.

Proceeding on this plan we find the following result. Let the annuity be payable $n$ times a year, let the age be $k$, and let $a_k$ be the number in the table alive at that age, $A_k$ being the tabular yearly annuity. Also let

$$\beta=\frac{n-1}{2n}, \qquad \beta'=\frac{n^2-1}{72n^2}, \qquad \beta''=\frac{n^4-1}{180n^4}.$$

The value* of the annuity then is, accurately, so far as the suppositions are accurate ($r$ being the interest of £1 for one year)

$$\left(1+6\beta'\frac{r^2}{1+r}-(\beta'+\beta'')\frac{r^4}{(1+r)^2}\right)A_k$$
$$+\beta-(8\beta'+2\beta'')\frac{r}{1+r}-(\beta'+\beta'')\frac{r^2}{(1+r)^2}$$
$$-3\beta'\frac{a_{k-1}-a_{k+1}}{a_k}+(\beta'+\beta'')r\left(\frac{a_{k+1}}{a_k}+\frac{1}{1+r}\frac{a_{k-1}}{a_k}\right).$$

This expression contains a coefficient for $A_k$ depending ($n$ apart) on the rate of interest only; certain terms which depend on the

* As we give no demonstration, we should say that, besides obtaining this expression from demonstration, we have tried several verifications: the simplest is that in which $a_k$ is a constant, and $A_k=1:r$, which gives the perpetual annuity.

rate of interest only; another depending on the mortality only; and others depending jointly on the rate of interest and the mortality. To form an idea of the utmost value of these corrections, let us take the extreme case of an annuity payable momently (or $n$ infinite) and a high rate of interest, 10 per cent. We have then

$$\beta=\frac{1}{2}, \qquad \beta'=\frac{1}{72}, \qquad \beta''=\frac{1}{180};$$

and the value of the annuity is

$$1{\cdot}00076\,A_k+{\cdot}48873-{\cdot}03989\frac{a_{k-1}}{a_k}+{\cdot}04361\frac{a_{k+1}}{a_k},$$

$A_k+{\cdot}5$ being the common approximation.

We may say then that the common approximation is generally correct to the first place of decimals, but by no means to the second or third. Now, it may fairly be insisted, looking at the manner in which such calculations are usually conducted, that the second place at least in the decimal part of an annuity shall be made as correct as can be conveniently done; not that this is, generally speaking, of very great consequence, but because there are other things by the dozen of no greater consequence, which are always attended to.

3. If in the preceding money make no interest, $A_k$ becomes $(a_{k+1}+a_{k+2}+\ldots)$ divided by $a_k$, and the annuity represents the mean duration of life, which is therefore ($n$ being infinite)

$$\frac{a_{k+1}+a_{k+2}+\ldots}{a_k}+\frac{1}{2}-\frac{1}{24}\frac{a_{k-1}-a_{k+1}}{a_k},$$

the first two terms are those from which the mean duration is usually deduced: the third is substantially the correction which we have elsewhere* proposed.

4. An annuity for a term of years must be computed in the usual way from the corrected life annuities.

5. No tables have been published for the mean duration of pairs of lives. This defect may be approximately supplied (see the work last cited, Appendix, p. xxx.), but much better from Mr. Jones's Tables of Annuities at 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, and 5 per cent. If these be called $A_6$, $A_7$, $A_8$, $A_9$, and $A_{10}$, the approximate formula is

$$105(2A_6+9A_8)+126A_{10}-80(9A_7+7A_9)+{\cdot}5.$$

Example 1. Required the mean duration of a life aged 45 (Carlisle Table).

* Cab. Cyclop. Probabilities and Life Contingencies, p. 163.

| | | |
|---|---|---|
| $(2A_6+9A_8=158\cdot671)\times105=$ | $16660\cdot455$ | $15942\cdot885$ |
| $(A_{10}=12\cdot648)\times126=$ | $1593\cdot648$ | $1521\cdot828$ |
| | $18254\cdot103$ | $17464\cdot713$ |
| $(9A_7+7A_9=227\cdot870)\times80=$ | $18229\cdot600$ | $17440\cdot160$ |
| | $24\cdot503$ | $24\cdot553$ |
| | $\cdot5$ | $\cdot5$ |
| | $25\cdot003$ | $25\cdot053$ |
| Correct answer | $24\cdot46$ | |

Example 2. Supposing male and female life not to be of different value, what is the average duration of a marriage between parties aged 30 and 25 ?

Answer, 25 years, calculated as above on the right.

Example 3. What is the average age at which a son of 7 years succeeds to the estate of a father of 37, and how long on the average does he enjoy it?

The first part of the question means the average age of those sons who do succeed, without any reference to those who do not survive their fathers; and since at 37 the mean duration of life is 30 years, the answer is that such sons succeed, one age with another, at the age of 37. The last part of the question cannot be directly solved by existing tables, if it refer to the actual average period of enjoyment of all those persons who live to come into possession; but it must first be found how many years of enjoyment those sons have, one with another, on the supposition that the actual years of enjoyment are divided among the whole, both those who live to come into possession and those who do not. If $L_x$ be the mean duration of $(x)$, and $L_{x,y}$ that of $(x)$ and $(y)$ jointly, then $L_7-L_{7,37}$ is *approximately* the mean period by which a son of 7 years survives a father of 37. Or, if $L_x$ and $L_{x,y}$ represent the values of the annuities when money makes no interest, $L_x-L_{x,y}+\cdot5$ is the mean period of survivorship; then $L_{x,y}$ is calculated by the above method, omitting the $\cdot5$, and $L_x+\cdot5$ is the mean duration of $(x)$ in the table. Calculate $L_{7,37}$, which gives $27\cdot62$, and $L_7$ being $50\cdot30$, we have $22\cdot68$ (say 23) years for the average term of enjoyment. That is to say, if 1000 sons, each aged 7, and the father of each being aged 37, were to agree to club their future years of possession in such manner that all (or their executors) should take equal shares, whether they lived to come into possession or not, the years of enjoyment of those who do come into possession would yield to each of the 1000 sons

23 years a-piece, or thereabouts; and of course the average term of those who *do* come into possession, nothing being supposed given up, is considerably larger than this; and an approximation (and only an approximation, though very plausible-looking reasoning might be put together to make it appear an exact method) may be made to it as follows: Divide this term of 23 years by the chance which a son of 7 years has of surviving a father aged 37. In the table of chances of survivorship, it appears that ·7837 is the chance of (7) surviving (37), and $23 \div \cdot 7837 = 29$, which is about the number of years required. This approximation always gives the result too great, perhaps by a year or two; but rough as the method may be, it is, as far as we know, the first which has been proposed—the want of that most essential preliminary, tables of mean survivorship of two lives, having prevented the subject from being considered.

Example 4. What would be the joint duration of twins if they were assumed to be, at birth, lives of the same value as other infants? The calculation gives 21 years.

Example 5. What is the mean duration of the survivor of (25) and (30)? This must be found, approximately, from the formula $L_{25} + L_{30} - L_{25,\,30} + \cdot 5$, or $37\cdot 4 + 33\cdot 8 - 24\cdot 6 + \cdot 5$, or 47 years.

Example 6. What is the mean joint duration of the three lives (7), (24), and (40)? Take the two elder lives and calculate their mean joint duration, which is found to be 26·5. Find a life which has this mean duration, which is (42), and find the mean joint duration of (7) and (42) or 27 years. This result is absurd, since the mean duration of (42) itself is not quite 27 years; it arises from the error of the process. In the present instance the mean duration of (7) is upwards of 50 years, so that the joint duration of (7) and (42) is very little less than that of (42) itself; so little less, that the error incidental to the process (which is generally in excess) more than makes the difference.

Also the annuities used in the application of this rule must be tolerably exact to three decimals; and if there be any doubt about the interpolated annuities, recourse must be had to those derived from Barrett's tables.

6. Supposing an individual $(x)$ to lay by £1 every year, what may he reasonably count upon having at his death, supposing that he improves the money at compound interest?

Supposing £1 to be laid by now, the question is the same as, how much should a premium of £1 assure at his death? and the answer is;—$N_{x-1}$ divided by $M_x$ in the table, or $(1+r)(1+A)$

divided by $1-r\text{A}$ from the common table, A being the value of a life annuity of £1. We mention this here because there is a table in Mr. Morgan's "Treatise on Assurances" (Table III., described in p. 54, and note V.) which was copied in part into the "Penny Cyclopædia," as being what its name would import, "the sum to which an annuity forborne and improved will amount on the extinction of a given life." Mr. Morgan's table gives the answer to the following question; if every one at his death were to receive his annuity (not a premium; exclude therefore the immediate payment) improved to its amount at the beginning of the year, what would persons of the same age at commencement receive one with another, from all the different amounts which they actually do receive? In our notation it is a calculation of the value of

$$\frac{a_{x+1}-a_{x+2}}{a_x}+\frac{a_{x+2}-a_{x+3}}{a_x}(1+\overline{1+r})+\frac{a_{x+3}-a_{x+4}}{a_x}(1+\overline{1+r}+\overline{1+r}^2)+\ldots$$

or,
$$\frac{a_{x+1}}{a_x}+\frac{a_{x+2}}{a_x}(1+r)+\frac{a_{x+3}}{a_x}(1+r)^2+\ldots.$$

This table may therefore serve (at least by the Northampton table at 4 per cent.) to compare the following cases. A and B are both of the same age: A is to assure his life immediately at a premium of £1, and B will in a year lay by £1, and the same at the end of every subsequent year. Supposing each to get the average of his class at death, what will they severally get? To bring both accounts up to the end of the year of death, at which the common assurance is supposed to be paid, a year's interest must be added to the result in Mr. Morgan's table. The following are the average sums obtained in the two cases, at different ages:—

| | A | B |
|---|---|---|
| 20 | 49·4 | 98·3 |
| 25 | 44·7 | 82·2 |
| 30 | 40·2 | 68·6 |
| 35 | 35·7 | 56·8 |
| 40 | 31·3 | 46·7 |
| 45 | 27·2 | 38·0 |
| 50 | 23·2 | 30·7 |
| 55 | 19·7 | 24·5 |
| 60 | 17·0 | 19·2 |

Supposing the Northampton table and 4 per cent. to be facts (we have chosen this table and rate because the only deduced table of the kind we are considering is Mr. Morgan's), it thus appears that the office gets from persons aged 20, one with another, in premiums and interest, £98·3 and the first premium with its

interest, from each person, for every £49·4 assured. But many of these persons receive each his £49·4 long before he has earned it by his premiums, and the advances thus made by the office must be repaid with interest by those who live longest.

7. The effect of compound interest, as distinguished from simple, is not very easily made apparent, and is not of great importance, since simple interest, as applied to annuities, is really a fiction, not more worthy of attention than would be the transactions of a man who, having a number of sovereigns in his hand, and others coming in, should invent a rule for separating some from the rest, and locking them up in a cash-box, instead of investing them. Some writers have argued that the *theory* of annuities at simple interest must be wrong, inasmuch as its consequence is, that an infinite sum is necessary to pay a perpetual annuity; the simple truth being, that if any one were to carry that theory into practice, he would find that the prediction of theory was verified; namely, that no sum, however great, would enable him to pay a perpetual annuity. According to the theory, the present value of an annuity (the rate of interest being $r$ per pound) of £1 for $n$ years is

$$\frac{1}{1+r}+\frac{1}{1+2r}+\frac{1}{1+3r}+\dots+\frac{1}{1+nr}.$$

If $n$ be considerable, the labour of summation will be materially lessened by using the following approximation.

$$\text{Let} \quad \frac{1}{1+r}=\mathrm{P}_1, \qquad \text{and } \frac{1}{1+nr}=\mathrm{P}_n$$

the value of the preceding series is nearly (the logarithm used being the common one)

$$\frac{2\cdot3025851}{r}\log\frac{1+nr}{1+r}+\frac{1}{2}(\mathrm{P}_1+\mathrm{P}_n)+\frac{r}{12}(\mathrm{P}_1{}^2-\mathrm{P}_n{}^2)-\frac{r^3}{120}(\mathrm{P}_1{}^4-\mathrm{P}_n{}^4).$$

To try this approximation, suppose $r$ as great as ·1 and $n=10$. We have then $1:r=10$, $nr=1$ (the powers of $\mathrm{P}_1$ may be taken from the table of present values of £1).

$$\begin{array}{lcr}
\left(\log\frac{2}{1\cdot1}=\cdot2596373\right)\times\frac{2\cdot3025851}{\cdot1} & = & 5\cdot978370 \\
\frac{1}{2}(\cdot909091+\cdot5) & = & \cdot704546 \\
\frac{\cdot1}{12}(\cdot826446-\cdot25) & = & \cdot004804 \\
=\frac{(\cdot1)^3}{120}(\cdot683013-\cdot0625) & =- & \cdot000005 \\
 & & \overline{6\cdot687715}
\end{array}$$

| | | Reciprocal. |
|---|---|---|
| *Verification.* | 1·1 | ·90909091 |
| | 1·2 | ·83333333 |
| | 1·3 | ·76923077 |
| | 1·4 | ·71428571 |
| | 1·5 | ·66666667 |
| | 1·6 | ·62500000 |
| | 1·7 | ·58823529 |
| | 1·8 | ·55555556 |
| | 1·9 | ·52631579 |
| | 2·0 | ·50000000 |
| | | 6·68771403 |
| | Approx. | 6·687715 |
| | Error | 0·000001 |

The approximative process would not be more difficult, nor less exact, if the number of terms were a million.

8. A more practical question than the last is derived from the following considerations. It is well known that individuals rarely or never make compound interest of what they lay by; while it is possible that many lay by a sum which increases from year to year equally or nearly so. We have seen (in 6) that an assurance office seems to offer a greater moral benefit to the old than the young; inasmuch as the latter, in making the office their trustee, seem to give up a more brilliant average prospect. But this is on the supposition that compound interest is made: let us now suppose that the *office* makes compound interest, while the *individual* would only lay by an increasing annual sum. We now introduce the following table. (See p. 145.)

This table is nothing but the table described in our last paper, on the supposition that money makes no interest, so that all results depend merely on the average duration of life. This table, as we shall proceed to show, concerns persons in general more almost than any other of the kind. All persons speculate, more or less, on what they can do with their incomes during their lives, without thinking of interest; that is, they attempt to form notions which this table would enable them to form justly.

Example 1. A person aged $x$ is to lay by £1 at the end of every year from this time. What will the savings amount to at his death, one person with another?

*Ans.* $\frac{N_x}{D_x}$; if $x=20$, $\frac{249432}{6090}=£40·96$.

| Age. | D. | N. | S. | Age. | D. | N. | S. |
|---|---|---|---|---|---|---|---|
| 0 | 10000 | 382213 | 12603644 | 53 | 4211 | 77784 | 967464 |
| 1 | 8461 | 373752 | 12221431 | 54 | 4143 | 73641 | 889680 |
| 2 | 7779 | 365973 | 11847679 | 55 | 4073 | 69568 | 816039 |
| 3 | 7274 | 358699 | 11481706 | 56 | 4000 | 65568 | 746471 |
| 4 | 6998 | 351701 | 11123007 | 57 | 3924 | 61644 | 680903 |
| 5 | 6797 | 344904 | 10771306 | 58 | 3842 | 57802 | 619259 |
| 6 | 6676 | 338228 | 10426402 | 59 | 3749 | 54053 | 561457 |
| 7 | 6594 | 331634 | 10088174 | 60 | 3643 | 50410 | 507404 |
| 8 | 6536 | 325098 | 9756540 | 61 | 3521 | 46889 | 456994 |
| 9 | 6493 | 318605 | 9431442 | 62 | 3395 | 43494 | 410105 |
| 10 | 6460 | 312145 | 9112837 | 63 | 3268 | 40226 | 366611 |
| 11 | 6431 | 305714 | 8800692 | 64 | 3143 | 37083 | 326385 |
| 12 | 6400 | 299314 | 8494978 | 65 | 3018 | 34065 | 289302 |
| 13 | 6368 | 292946 | 8195664 | 66 | 2894 | 31171 | 255237 |
| 14 | 6335 | 286611 | 7902718 | 67 | 2771 | 28400 | 224066 |
| 15 | 6300 | 280311 | 7616107 | 68 | 2648 | 25752 | 195666 |
| 16 | 6261 | 274050 | 7335796 | 69 | 2525 | 23227 | 169914 |
| 17 | 6219 | 267831 | 7061746 | 70 | 2401 | 20826 | 146687 |
| 18 | 6176 | 261655 | 6793915 | 71 | 2277 | 18549 | 125861 |
| 19 | 6133 | 255522 | 6532260 | 72 | 2143 | 16406 | 107312 |
| 20 | 6090 | 249432 | 6276738 | 73 | 1997 | 14409 | 90906 |
| 21 | 6047 | 243385 | 6027306 | 74 | 1841 | 12568 | 76497 |
| 22 | 6005 | 237380 | 5783921 | 75 | 1675 | 10893 | 63929 |
| 23 | 5963 | 231417 | 5546541 | 76 | 1515 | 9378 | 53036 |
| 24 | 5921 | 225496 | 5315124 | 77 | 1359 | 8019 | 43658 |
| 25 | 5879 | 219617 | 5089628 | 78 | 1213 | 6806 | 35639 |
| 26 | 5836 | 213781 | 4870011 | 79 | 1081 | 5725 | 28833 |
| 27 | 5793 | 207988 | 4656230 | 80 | 953 | 4772 | 23108 |
| 28 | 5748 | 202240 | 4448242 | 81 | 837 | 3935 | 18336 |
| 29 | 5698 | 196542 | 4246002 | 82 | 725 | 3210 | 14401 |
| 30 | 5642 | 190900 | 4049460 | 83 | 623 | 2587 | 11191 |
| 31 | 5585 | 185315 | 3858560 | 84 | 529 | 2058 | 8604 |
| 32 | 5528 | 179787 | 3673245 | 85 | 445 | 1613 | 6546 |
| 33 | 5472 | 174315 | 3493458 | 86 | 367 | 1246 | 4933 |
| 34 | 5417 | 168898 | 3319143 | 87 | 296 | 950 | 3687 |
| 35 | 5362 | 163536 | 3150245 | 88 | 232 | 718 | 2737 |
| 36 | 5307 | 158229 | 2986709 | 89 | 181 | 537 | 2019 |
| 37 | 5251 | 152978 | 2828480 | 90 | 142 | 395 | 1482 |
| 38 | 5194 | 147784 | 2675502 | 91 | 105 | 290 | 1087 |
| 39 | 5136 | 142648 | 2527718 | 92 | 75 | 215 | 797 |
| 40 | 5075 | 137573 | 2385070 | 93 | 54 | 161 | 582 |
| 41 | 5009 | 132564 | 2247497 | 94 | 40 | 121 | 421 |
| 42 | 4940 | 127624 | 2114933 | 95 | 30 | 91 | 300 |
| 43 | 4869 | 122755 | 1987309 | 96 | 23 | 68 | 209 |
| 44 | 4798 | 117957 | 1864554 | 97 | 18 | 50 | 141 |
| 45 | 4727 | 113230 | 1746597 | 98 | 14 | 36 | 91 |
| 46 | 4657 | 108573 | 1633367 | 99 | 11 | 25 | 55 |
| 47 | 4588 | 103985 | 1524794 | 100 | 9 | 16 | 30 |
| 48 | 4521 | 99464 | 1420809 | 101 | 7 | 9 | 14 |
| 49 | 4458 | 95006 | 1321345 | 102 | 5 | 4 | 5 |
| 50 | 4397 | 90609 | 1226339 | 103 | 3 | 1 | 1 |
| 51 | 4338 | 86271 | 1135730 | 104 | 1 | 0 | 0 |
| 52 | 4276 | 81995 | 1049459 | | | | |

Example 2. A person aged $x$ is to lay by £1 for $n$ years, if he live so long: what will be the average amount?

$$Ans.\quad \frac{N_x-N_{x+n}}{D_x};\ \text{if } x=20,\ y=10, \frac{249432-190900}{6090}=£9{\cdot}6.$$

Example 3. What should a person aged $x$ lay by now and at the end of every year, that, one such person with another, there may be £1 at death?

$$Ans.\quad \frac{D_x}{N_{x-1}};\quad x=20,\quad \frac{6090}{255522}=£{\cdot}024,\ \text{say } 0s.\ 6d.$$

Example 4. A person aged $x$ is to lay by $a$ at the end of a year, $a+h$, $a+2h$, &c., at the end of 2, 3, &c., years; what will be the average savings of such persons, one with another?

$$Ans.\quad a\frac{N_x}{D_x}+h\frac{S_{x+1}}{D_x}.$$

This problem represents better than any other in our opinion, the *best* supposition that can be made as to what any individual can do who attempts to be his own life-assurer. If $x=20$, we have $41a+990h$ for the average required.

Suppose then an individual (A) in this way steadily to invest £1 every year, with three per cent. simple interest upon the preceding investments, so that $a=1$, $h={\cdot}03$, and the preceding is £71; such individuals, one with another, will have £71 at death; that is, some will have next to nothing, a majority less (many of them much less) than £71, and a minority more, a few of them much more. Another individual (B) goes with £1 down and £1 yearly to an office which charges the Carlisle premiums at 3 per cent., increased by 25 per cent. for management and fluctuation. Such an office, returning profits, will pay, let us say £120 for every £100 assured, to its customers, one with another. If this be the case it will pay the preceding individual £64 at his death. Let us now see how the parties stand.

| A | B |
|---|---|
| has been gambling, setting a remote chance of more than the office would give against a great risk of leaving next to nothing, or at any rate very much less than the policy. He has had all the trouble of investing his | has exchanged chance for certainty at a very moderate sacrifice, owing to the office being able to gain compound interest. He has no trouble in investing, beyond that of a walk once a year to the office, and is very |

savings, all the risk of being led to engage in some foolish speculation* which should lose them, and if neither his industry nor his prudence should fall short, all the anxiety of leaving too little behind him for many years. *If* he succeed, he may succeed very signally, and he may have the compound interest of his savings, if he know how to make it; but it is much more than an even chance that he does not do nearly so well as B.

nearly rid of temptation to change his investment by knowing that he can only do it at a loss, and by having felt the comfort of knowing himself secure. He has, from the very beginning, a certainty of having made sure of his average, a state of mind highly conducive to health. His success is moderate, but certain; and it never surprises him if, any day of any week, he is solicited to aid in saving from starvation the widow and children of A.

9. We shall end this paper with the consideration of a question which is of frequent occurrence, namely, the borrowing of money to be paid by annuity. A person aged $x$ (the annuity on his life being worth $A_x$ year's purchase) borrows £1, for which he is to pay a life annuity. This life annuity ought to be $1 : A_x$ of a pound. Now if $p_x$ be the premium of assurance for £1 at the required age, we have, $r$ being the interest of £1 for one year,

$$\frac{1}{1+A_x}-\frac{r}{1+r}=p_x, \qquad \text{or } \frac{1}{A_x}=\frac{r+(1+r)p_x}{1-(1+r)p_x}.$$

First, suppose an assurance office to exist, charging the absolute premium of the tables, and nothing more. Let the lender insure the borrower's life, and take from him a yearly payment equivalent both to the interest of the sum lent and the premium. Remember that the lender pays the premium at the beginning of the year, while the borrower's annuity payment does not become due till the end, whence the borrower must pay interest for each premium advanced. This he does (with the interest of the loan) as long as he lives, and, at the end of the year of death, the lender claims from the office: 1, the £1 lent; 2, a year's interest on it, just due; 3, the premium last paid; 4, a year's interest due on it. Let S be the sum assured; then S must be made up of 1, $r$, $p_x$S (its premium), and $rp_x$S a year's interest. Hence

$$S=1+r+p_x(1+r)S, \quad \text{or } S=\frac{1+r}{1-p_x(1+r)}=\frac{1}{v-p_x}.$$

* In particular, the temptation of spending the *interest* of his savings in making a greater appearance, a folly which has prevented many a competency from being acquired.

which is the sum the lender must assure, to keep himself clear of all loss. Now the borrower must pay at the end of each year, 1, the interest just due on £1; 2, the preceding premium advanced for him a year before, with interest, so that his annuity must be

$$r+\frac{(1+r)^2}{1-p_x(1+r)}.p_x, \quad \text{or } \frac{r+(1+r)p_x}{1-(1+r)p_x}, \quad \text{or. } \frac{1}{v-p_x}-1,$$

as found before. And this last process* is true, whether the office charges the tabular premium, or makes an addition.

Let us now examine how far the existence of assurance offices affects the position of a borrower. Formerly, in borrowing on annuity, the usury laws made the position of the lender hardly respectable, for his risk (as no one individual could have many transactions of the kind, so as to secure an average) required what might either be called a high rate of mortality or a high rate of interest, at the pleasure of the lender; for the price which supposed 4 per cent., deduced from one table, might be the price at 8 per cent. deduced from another. It is certain that, with the money-lenders, 10 per cent. on the Northampton table was not uncommon. The life of a man of 35 can be respectably insured for £2. 10*s*. per £100, or £·025 per £1. If he borrow at 5 per cent. in the preceding manner, we have then $r=\cdot05$, $p_x=\cdot025$, and £·0783 is the value of the preceding expression, say, £78. 6*s*.* for £1,000. Now, if this man had gone to a money-lender who asked 10 per cent., even taking the Carlisle table (which is favourable in this case to the borrower), he must have paid an annuity of £116. 11*s*. for the same accommodation. And in many other ways it might be shown that the existence of assurance-offices is a most valuable protection, not as enabling persons to borrow (an objection we have before now heard urged against them), for borrowing was always practicable, but as enabling them to borrow on reasonable terms, and of respectable people; which the usury laws had rendered almost impracticable.

This transaction must be arranged as follows, when it is desired to pay off the loan in $n$ years. Let $p$ be the premium for assuring £1 for $n$ years, $k$ that for endowing the debtor with £1, if he live over $n$ years. The lender must then (the loan being supposed £1) insure the debtor's life for $\frac{1+k}{v-p}$, and must also pay a premium

* As far as I know, this exact method is due to Mr. Griffith Davies, and the first proof of it is in Mr. Jones's work (p. 189).

for £1, to be paid by the office if the debtor survive $n$ years. This premium to the office will therefore be

$$\frac{1+k}{v-p}p+k, \text{ or } \frac{p+vk}{v-p}.$$

And the debtor's annuity is $\dfrac{1+k}{v-p}-1$, the first payment being due in a year, and the last at the end of the $n$ years, if he live so long.

A. DE MORGAN.

*University College, June* 21, 1841.

*** Mr. Peter Gray has favoured us with the following errata in our reprint of the first Number:—

| | | | | | |
|---|---|---|---|---|---|
| Page 334, | line 25, | for | $+m-1)\beta$, | read | $+(m-1)\beta$. |
| ,, 335, | ,, 6, | ,, | $\theta R_{x+1,} -\nu\varpi R_{x,\ +1}$, | ,, | $\theta R_{x+1,w}-\nu\varpi R_{x,w+1}$. |
| ,, 338, | ,, last, | ,, | comes, | ,, | becomes. |
| ,, 340, | ,, 18, | ,, | $a(hn=a)$, | ,, | $a$, $(hn=a)$. |
| ,, ,, | ,, 20, | ,, | *dccreasing*, | ,, | *decreasing*. |
| ,, 341, | ,, 35, | ,, | and, | ,, | into. |
| ,, 344, | ,, 14, | ,, | $\mu-1$, | ,, | $\mu=1$. |
| ,, 345, | ,, 26, | ,, | portion, | ,, | proportion. |
| ,, 348, | ,, 6, | ,, | Y, | ,, | $Y\varpi$. |

## *On the Construction of Tables by the Method of Differences. By* Peter Gray, F.R.A.S., *Honorary Member of the Institute of Actuaries.*

### Section II.—*On the Calculus of Finite Differences.*

(50). The Calculus of Differences is a somewhat extensive subject, but it is not necessary now to go far into it, or to occupy ourselves with its application to other than the simplest class of functions, namely, rational, algebraical, and integer functions. The theorems we require might have been left to be deduced as occasion for them should arise; but it has been thought better—and the arrangement will certainly be more convenient for reference—to bring the greater portion of them together in this place.

(51). In this calculus functions of a variable, as $x$, are denoted by the letters $u$, $v$, &c., with the variable attached. Thus, $u_x$, $v_x$, &c., denote functions of $x$, and $u_{x+n}$, $v_{x+n}$, &c., denote the same functions, respectively, when $x$ in them is changed into $x+n$. Also, when $x$ takes a particular value, 10 for example, they will be denoted by $u_{10}$, $v_{10}$, &c.